The work input to a granular material

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INTRODUCTION

Some recent theoretical models for soils are based on hypotheses about the rate at which the input work to the soil is either stored or dissipated. It is therefore necessary to evaluate the rate of work input to the soil in terms of the stresses, strain rates and other variables. For a single-phase material the power input per unit volume is simply the product of the stresses and the strain rates; but this result does not apply for the two-phase material, such as a saturated soil, where both the stresses within the two phases and the velocities of the two materials will be different. The total power input per unit volume to a soil must therefore be derived by considering the rate at which all the forces on both the soil grains and the pore fluid do work.

The principle of effective stress as described by Terzaghi (1943) states that the mechanical behaviour of a soil is governed by the difference between the total stress and the pore pressure, this quantity being termed the effective stress. If it is accepted that the mechanical behaviour of a material reflects the storage and dissipation of the power input, then it would be expected that this power input for a soil would depend on effective, not total stress. Such a result was obtained by Schofield and Wroth (1968) where they showed that, for an infinitely slow process, the mechanical work input to a soil is given by the product of the effective stress and the strain. (The result was obtained only for the special case of the triaxial test, but could be readily extended to more general stress states.)

Alternatively, if there is no deformation of the soil skeleton, the work input derives solely from the loss of excess pore pressure as the pore fluid seeps through the soil skeleton. The power input per unit volume may be calculated for this case as the product of the excess pore pressure gradient and the artificial seepage velocity.

The above two special cases represent extremes of soil behaviour: in the first, the entire power input is associated with the deformation of the soil skeleton, and in the second, the entire input is due to the viscous flow of the pore fluid. In many cases of engineering importance however, the two processes of deformation and seepage occur simultaneously, the best

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known example of this being the consolidation of a soil. For the general case it must be determined whether the total power input is simply the sum of the two terms above (effective stress times deformation rate and excess pore pressure gradient times artificial seepage velocity) or whether there are additional power terms representing some coupling between the two processes of seepage and skeleton deformation.

Note that such a coupling between various terms in the expression for the input power is not to be confused with the inter-relation between seepage and skeleton deformation which arises from the conditions of compatibility and equilibrium. These latter relationships form the basis of conventional consolidation analysis.

ANALYSIS

In the following analysis the total power input per unit volume to a soil is derived under the simplifying assumption that both the individual soil grains and the pore fluid may be considered as incompressible; the analysis is therefore applicable to saturated soils only. Since the analysis is in terms of continuum mechanics (in which the microscopically non-homogeneous soil is replaced by an equivalent homogeneous continuum) the region under consideration must be sufficiently large so that the definitions of stress, strain and porosity are meaningful. It is assumed that there is no overall shear stress in the pore fluid on the scale of the element under consideration. However, in order that there should be any loss of excess pore pressure there must be shear stresses in the pore fluid on the scale of the grain size.

The analysis is carried out in terms of tensors, since this notation allows a convenient shorthand for the many components of the different variables. As only strain rates are considered (which may be treated as small strains) there is no loss of generality in the use of Cartesian tensors. The summation convention over a repeated index is used, and the symmetric and skew-symmetric parts of the tensor a_{ij} are given by $a_{(ij)}$ and $a_{[ij]}$ respectively. Partial differentiation with respect to spatial co-ordinate only is given by a comma notation; thus $\partial a/\partial x_i = a_{,i}$. Such differentials may be treated as tensors. The symbol δ_{ij} is the Kronecker delta ($\delta_{ij} = 1$, $i = j$; $\delta_{ij} = 0$, $i \neq j$).

Let the total stress on an element of soil be σ_{ij} and the total pore pressure be u, the effective stress σ_{ij}' is then defined as

$$\sigma_{ij}' = \sigma_{ij} - u\delta_{ij} \qquad \text{(1)}$$

If the porosity is n and a plane cutting through both particles and voids is considered, then the average stress within the grains is given by s_{ij}, where

$$\sigma_{ij} = nu\delta_{ij} + (1-n)s_{ij} \qquad \text{(2)}$$

Denoting the densities of the soil grains and the pore fluid by $\rho^{(s)}$ and $\rho^{(w)}$ respectively, then the overall density ρ is given by

$$\rho = n\rho^{(w)} + (1-n)\rho^{(s)} \qquad \text{(3)}$$

If g_i is the gravitational acceleration the variation of excess pore pressure u' may be defined as

$$u_{,i}' = u_{,i} - \rho^{(w)}g_i \qquad \text{(4)}$$

The velocity of the soil skeleton is given by v_i and the strain by $\dot{\varepsilon}_{ij}$; the strain rate may then be defined as

$$\dot{\varepsilon}_{ij} = -v_{(j,i)} \qquad \text{(5)}$$

where the superposed dot indicates partial differentiation with respect to time, and the negative sign arises from a definition of compressive strains as positive. Denoting the actual mean velocity of the pore fluid by f_i, the artificial seepage velocity w_i (as used in conventional seepage calculations) is then defined by

$$w_i = n(f_i - v_i) \quad \ldots\ldots\ldots\ldots \quad (6)$$

The above equations define the various quantities which are of interest, and correspond excactly to the conventional definitions used in soil mechanics. The conditions of equilibrium and compatibility now impose certain relationships between these quantities. The equation of total stress equilibrium, taking compressive stress as positive, may be written as

$$-\sigma_{ij,j} + \rho g_i = 0 \quad \ldots\ldots\ldots\ldots \quad (7)$$

Since both the pore fluid and the individual soil grains are considered as incompressible the total flow of material into an arbitrary volume V fixed in space must be zero. If A is the surface bounding V, and ν_j a unit vector in the direction of the outward normal to an element of this surface dA, then this condition may be written

$$\int_A [nf_j + (1-n)v_j]\nu_j dA = 0$$

Substituting the definition of artificial velocity (equation (6)) and making use of the divergence theorem of Gauss[1]

$$\int_V (w_j + v_j)_{,j} dV = 0$$

Carrying out the differentiation and noting that since V is arbitrary the integrand must always be zero

$$w_{j,j} + v_{j,j} = 0 \quad \ldots\ldots\ldots\ldots \quad (8)$$

which is the final form of the compatibility condition for incompressible soil grains and pore fluid.

The power input to any arbitrary volume fixed in space is obtained from the sum of the products of the various forces (body forces and surface tractions) acting on the material with their respective velocities. If the average power input per unit volume within V is L, then summation of these terms gives

$$\int_V L\,dV = -\int_A [nu\delta_{ij}f_i + (1-n)s_{ij}v_i]\nu_j dA + \int_V [\rho^{(w)}ng_if_i + \rho^{(s)}(1-n)g_iv_i]dV$$

where the negative sign arises from the compressive positive convention for the stresses. Substituting equations (2) and (3) gives

$$\int_V L\,dV = -\int_A (nu\delta_{ij}f_i - nu\delta_{ij}v_i + \sigma_{ij}v_i)\nu_j dA + \int_V (\rho^{(w)}ng_if_i - \rho^{(w)}ng_iv_i + \rho g_iv_i)dV$$

Substituting the definition of artificial velocity (equation (6)) and then noting that $\delta_{ij}w_i = w_j$, the divergence theorem of Gauss may then be applied to give

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[1] The divergence theorem of Gauss states that if y is an analytical function of x_i, ie $y = y(x_i)$, then, using the terminology defined above,

$$\int_A y\nu_j dA = \int_V y_{,j} dV$$

$$\int_V L dV = \int_V -(uw_j + \sigma_{ij}v_i)_{,j} + (\rho^{(w)}g_i w_i + \rho g_i v_i) dV$$

Again noting that since V is arbitrary this may be written in its local form, expansion of the differential then results in

$$L = -u_{,j}w_j - uw_{j,j} - \sigma_{ij,j}v_i - \sigma_{ij}v_{i,j} + \rho^{(w)}g_i w_i + \rho g_i v_i$$

Altering the dummy indices of the first term allows substitution of the definition of excess pore pressure gradient (4). Substitution of the compatibility condition (8) then allows rearrangement to

$$L = -u_{,i}'w_i - \sigma_{ij}v_{i,j} + uv_{j,j} - \sigma_{ij,j}v_i + \rho g_i v_i$$

The last two terms may be eliminated by use of the equilibrium condition (7). Using the relationship $v_j = \delta_{ij}v_i$

$$L = -u_{,i}'w_i - \sigma_{ij}v_{i,j} + u\delta_{ij}v_{i,j}$$

Substituting the definition of effective stress (1)

$$L = -u_{,i}w_i - \sigma_{ij}'v_{i,j}$$

Note that because of the symmetry of σ_{ij}', $\sigma_{ij}'v_{[i,j]} = 0$, so $\sigma_{ij}'v_{i,j} = \sigma_{ij}'v_{(i,j)}$. Making use of the symmetry of $v_{(i,j)}$ to interchange indices, the substitution of the definition of strain rate (5) then yields the final result

$$L = \sigma_{ij}'\dot{\varepsilon}_{ij} - u_{,i}'w_i$$

DISCUSSION

It has been proven that there are no terms in the power input expression other than the two discussed above. The simple expression for the power input per unit volume therefore continues to apply for the case of finite deformation rate combined with seepage, the two terms being the product of the effective stress with the strain rate and the (negative) excess pore pressure gradient with the artificial seepage velocity. (The negative sign results simply from the sign convention for excess pore pressure gradient.)

This result may be used to give a new interpretation of the principle of effective stress. If the Terzaghi definition of effective stress is adopted it is observed that the total rate of work input per unit volume to the soil is given by the two terms $(\sigma_{ij}'\dot{\varepsilon}_{ij})$ and $(-u_{,i}'w_i)$. Clearly these may be interpreted as the rates of work input per unit volume to the soil skeleton and to the pore fluid respectively, and there is no coupling between the two processes of skeleton deformation and seepage. This result may be inverted to state that if there is no coupling between the work input to the soil skeleton and to the pore fluid, then the power input per unit volume to the soil skeleton is given at all times by the product of the effective stress with the strain rate. If it is further stated that the mechanical behaviour is simply a reflection of the way in which work is stored and dissipated, then if the processes of skeleton deformation and seepage are uncoupled, the mechanical behaviour of the skeleton will depend on the effective stress as defined by Terzaghi.

Although this gives an alternative interpretation of the principle of effective stress in terms of continuum mechanics, no statement is made here about whether soil would be expected to obey the principle, and hence show the uncoupling of the work terms. Any justification of the principle of effective stress for soils still rests on the arguments of particulate mechanics (notably Bishop, 1959) and the extensive body of experimental support for the theory. The present analysis offers, however, a new interpretation in terms of continuum mechanics, in that the principle of effective stress is seen as a principle of the independence of the mechanical work input to the soil skeleton and to the pore fluid.

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